Nucleon separation energy — A semi-empirical estimate of the *nucleon separation energy*, S_N may be easily made, for

$$S_N = B(A) - B(A-1) = \frac{dB}{dA}$$

But, $$\frac{d(B/A)}{dA} = \frac{1}{A}\frac{dB}{dA} - B\frac{1}{A^2}$$

$$\therefore\ S_N = \frac{dB}{dA} = A\frac{d(B/A)}{dA} + \frac{B}{A} \tag{8.2.18}$$

and can be evaluated directly by a measurement of the slope of B/A vs. A plot (Fig. 8.5).

The liquid drop model has other important applications: theory of compound nucleus and the phenomenon of nuclear fission, α-disintegration, β-disintegration energy of mirror nuclei.

Shell model

The large binding energy of He-nucleus (α-particle) suggests that 2 protons and 2 neutrons form a stable nuclear configuration. Taking the clue from the chemical stability of closed electron sub-shells and shells in atoms, the physicists enquired if nucleons too form similar closed sub-shells and shells in nuclei, i.e., if protons and neutrons in a nucleus are also arranged in some type of a shell structure. This idea of Elsasser eventually culminated in the development of the **nuclear shell model**.

8.3 Points in favour of the shell model

There exists a number of points in favour of the shell model of the nucleus and the following are worth noting.

1. Just as inert gases, with 2, 10, 18, 36, 54, $\cdots$ electrons, having closed shells show high chemical stability, nuclei with 2, 8, 20, 50, 82 and 126 nucleons—the so-called **magic numbers** — of the same kind (either proton or neutron) are particularly stable. The binding energy is found to be unusually high implying high stability which is reflected in high abundance of isotopes with these proton numbers and isotones with these neutron numbers. Nuclei both with Z and N = each a magic number, are said to be *doubly magic*.

2. The number of stable isotopes (Z = const.) and isotones (N = const.) is larger with respective number of protons and neutrons equal to either of the magic numbers: e.g., Sn ($Z = 50$) has 10 stable isotopes, Ca ($Z = 20$) has 6; the biggest group of isotone is at $N = 82$, then at $N = 50$ and $N = 20$.

3. The three naturally occurring radioactive series decay to the stable end product $^{208}_{82}$Pb with $Z = 82$ and $N = 126$ indicating extra stable configuration of magic nuclei.

4. The neutron absorption cross-section is low for nuclei with N = magic numbers like 50, 82 and 126, indicating reluctance of magic nuclei to accept extra neutrons in their completely filled shells.

5. Isotopes like ${}^{17}_{8}O$, ${}^{87}_{36}K$ and ${}^{137}_{54}Xe$ are spontaneous neutron emitters when excited by preceding β-decay. The isotopes have $N = 9$, 51 and 83 respectively. i.e., $N = (8+1)$, $(50+1)$ and $(82+1)$. One can interpret this loosely bound neutron as a *valence neutron* which the isotopes emit to assume some magic N-value for their stability.

6. Electric quadrupole moment Q of magic nuclei is zero indicating spherical symmetry of the nucleus for closed shells. When Z-value or N-value is gradually increased from one magic number to the next, Q increases from zero to a maximum and then decreases to zero at the next magic number.

7. The energy of α- or β-particles emitted by magic radioactive nuclei is larger.

All these experimental facts lend a strong support to the shell structure of nuclei.

8.4 Salient features (assumptions) of shell model

This model assumes that each nucleon stays in a well-defined quantum state. But, unlike the atom, the nucleus has *no obvious massive central body acting as fixed force centre of charge.*

In the shell model, therefore, each nucleon is considered as *a single particle that moves independently of others in the time-averaged field of the remaining* $(A-1)$ *nucleons* acting as a *core*, and is confined to its own orbit completing several revolutions before being disturbed by others by way of collisions. This implies that the mean free path before collisions of nucleons is much larger than the nuclear diameer. It amounts to assuming the interaction among the nucleons to be weak. This sounds paradoxical as nuclear matter is super-dense ($\sim 10^{17}$ kg/m^3) and experiments indicate that a nucleus is virtually opaque to any incident nucleon. This '*weak interaction paradox*' was saved by invoking Pauli's principle by Weisskopf. He argued that nucleons are fermions and by exclusion principle, no two neutrons and protons can stay in identical quantum state. Hence the experimentally expected strong interaction among nucleons in a nucleus cannot show itself since all the quantum states (low lying) into which the scattered nucleon after collision may go are already occupied.

In terms of Schrödinger's equation, each nucleon thus moves in the same potential $V(r)$ which may be taken as an *average harmonic oscillator potential* so that $V(r) = \frac{1}{2}kr^2$. Schrödinger equation then becomes

$$\left(-\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}kr^2\right)\psi = E\psi \qquad (8.4.1)$$

where M is the mass of the nucleon and E the energy eigenvalues.

The solution of equation (8.4.1) is given by :

$$E_n = \left(N + \tfrac{1}{2}\right)\hbar\omega \tag{8.4.2}$$

where N = oscillator quantum number $= 0, 1, 2, 3, \cdots$ so that in the harmonic oscillator model, all the energy eigen states are *equally spaced*. The wave function ψ has both angular (orbital) and the radial part.

Each nucleon is supposed to have an orbital angular momentum $|\vec{l}| = \sqrt{l(l+1)}\hbar$ where $l = 0, 1, 2, 3, \cdots$, the nuclear orbital quantum number. Another quantum number, very similar to but not the principal quantum number of electronic orbit, characterises the radial part of nuclear wave function and is symbolised by $n = 1, 2, 3, \cdots$. Each nucleon has also spin angular momentum $|\vec{s}| = \sqrt{s(s+1)}\hbar$ where $s = \frac{1}{2}$ and behaves as an independent particle subject to Pauli's principle that *no two identical nucleons can be in the same quantum state*. Again, as suggested by Mayer, Jensen and others, there is a *strong interaction (coupling) between the orbital and the intrinsic spin angular momenta* of each nucleon. The quantum mechanical rules for angular momenta dictate that total angular momentum $j\hbar$ formed by vector addition of orbital angular momentum $l\hbar$ and spin $s\hbar$ must be such that j (total angular momentum quantum number) is restricted to the following two values : $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$.

Thus a different energy is associated with each of the two j-levels and each nucleonic energy level with a given l splits into *two sub-levels*, except for $l = 0$, when j has only one value $\frac{1}{2}$. The level $j = l + \frac{1}{2}$ corresponds to $\vec{s}$ and $\vec{l}$ *parallel* ($\uparrow\uparrow$) to each other and $j = l - \frac{1}{2}$ to $\vec{s}$ and $\vec{l}$ *anti-parallel* ($\uparrow\downarrow$) to each other. Empirically, it is found that the nuclear energy level with higher j always lies below that with smaller j. So $j = l + \frac{1}{2}$ sub-level has a *lower energy* than $j = l - \frac{1}{2}$ sub-level, the former giving a more tightly bound nucleonic state. The separation between two sub-levels with a given l is rather large and increases rapidly with l. When $l \geq 4$, the sub-levels lie in different shells. The quantum number j of nucleus is obtained by $\sum j_i$, i.e., adding the j-values of the individual nucleons (similar to *jj-coupling* in atoms), subject however to the usual quantum conditions.

To designate the nucleonic states, spectroscopic notation of atomic physics is followed. Each sub-level can have a maximum of $(2j + 1)$ nucleons of the same kind, for a given j. So it can house $(2j + 1)$ protons and $(2j + 1)$ neutrons. Nucleons are designated with n-values followed by spectroscopic notation of l-values $(s, p, d, f, \cdots$ for $l = 0, 1, 2, 3, \cdots)$; the j-values are shown as subscript and the superscript gives the number of nucleons required to complete the sub-shell (Table 8.2).

For instance, for $l = 0$, $j = l + \frac{1}{2} = \frac{1}{2}$ and the number of nucleons in the level $= 2j + 1 = 2 \times \frac{1}{2} + 1 = 2$ and the state is designated along with n-value as $(1s_{\frac{1}{2}})^2$. Similarly, for $l = 1$, $j = l \pm s = l \pm \frac{1}{2} = \frac{3}{2}$ and $\frac{1}{2}$. The number of nucleons in the two sub-levels are thus $2 \times \frac{3}{2} + 1 = 4$ and $\left(2 \times \frac{1}{2} + 1\right) = 2$. The two sub-states are designated as $(1p_{\frac{3}{2}})^4$ and $(1p_{\frac{1}{2}})^2$ respectively. So the total number of nucleons in this level = $4+2=6$, giving the progressive total of 8 nucleons, and so it goes on. Thus we can predict the completed shells and the corresponding total number of nucleons.

The quantum number n, l and j and the designation of energy states in order of the energy and in accordance with the above notational scheme is shown in Table 8.2

Table 8.2 : *Nucleonic sub-shells and shells*

n	l	j	Designation and no. of p or n to fill sub-levels		Progessive Total
1	0	1/2	$(1s_{1/2})^2$	2	2
1	1	3/2	$(1p_{3/2})^4$		
1	1	1/2	$(1p_{1/2})^2$	6	8
1	2	5/2	$(1d_{5/2})^6$		
2	0	1/2	$(2s_{1/2})^2$		
1	2	3/2	$(1d_{3/2})^4$	12	20
1	3	7/2	$(1f_{7/2})^8$		
2	1	3/2	$(2p_{3/2})^4$		
1	3	5/2	$(1f_{5/2})^6$		
2	1	1/2	$(2p_{1/2})^2$		
1	4	9/2	$(1g_{9/2})^{10}$	30	50
1	4	7/2	$(1g_{7/2})^8$		
2	2	5/2	$(2d_{5/2})^6$		
2	2	3/2	$(2d_{3/2})^4$		
3	0	1/2	$(3s_{1/2})^2$		
1	5	11/2	$(1h_{11/2})^{12}$	32	82
1	5	9/2	$(1h_{9/2})^{10}$		
2	3	7/2	$(2f_{7/2})^8$		
2	3	5/2	$(2f_{5/2})^6$		
3	1	3/2	$(3p_{3/2})^4$		
3	1	1/2	$(3p_{1/2})^2$		
1	6	13/2	$(1i_{13/2})^{14}$	44	126

up to the magic number 126. The thick horizontal lines signal the *closure of a shell* where marked changes occur in the nucleonic binding energy.

The shell model with its spin-orbit coupling thus beautifully accounts for the magic nucleon numbers.

8.5 Shell model : infinite square well potential

We have already discussed in Art. 8.4, the salient features of the shell model using harmonic oscillator potential. With the risk of being to some extent repetitive, we shall discuss the model with square well potential.

In the study of nuclei, essentially a many-body problem, it is assumed that from the viewpoint of any one nucleon, the forces acting on it by all others may be represented by a potential well. This is known as the *shell theory of potential*. In this model the nucleons move independently in an average potential which we shall consider as *infinite square well potential* and the nucleus is visualized as consisting of filled shells having maximum number of neutrons and protons allowed by Pauli's principle. The unfilled shells contain the remaining number of neutrons and protons.

The eigenstates of a nucleon of mass M moving in a spherically symmetric potential $V(r)$ are given by the solutions of Schrodinger equation

$$\left[\nabla^2 + \frac{2M}{\hbar^2}(E - V(\vec{r}))\right]\psi(\vec{r}) = 0 \tag{8.5.1}$$

where E is the energy eigenvalues and ψ is given by

$$\psi_{nlm}(\vec{r}) = u_{nl}(\vec{r})Y_l^m(\theta,\phi). \tag{8.5.2}$$

$u_{nl}(\vec{r})$ satisfies the following equation in infinite square well potential :

$$-\frac{\hbar^2}{2M}\frac{1}{r}\frac{d^2}{dr^2}(ru_{nl}(\vec{r})) + \frac{\hbar^2}{2M}\frac{l(l+1)}{r^2}u_{nl}(\vec{r}) = Eu_{nl}(\vec{r})$$

with the boundary conditions : $u_{nl}(\vec{r})$ is finite at $r = 0$ and is zero at $r = R$, the nuclear radius.

Y_l^m are the *spherical harmonics*. The quantum number n is not the principal quantum number, but is associated with the number of nodes in the radial function. l is the orbital angular momentum quantum number taking up values $l = 0, 1, 2, \cdots$ etc. determines the parity of the state, which is odd (even) for odd (even) l.

For $l = 0$ (s-state), the solutions, finite at $r = 0$, are

$$u_{no}(\vec{r}) = \frac{\sin kr}{kr}; \quad E_{no} = \frac{\hbar^2k^2}{2M} \tag{8.5.3}$$

The boundary conditions at $r = R$ is satisfied by taking $k = k_n = n\pi/R$, $n = 1, 2, 3, \cdots$, and the corresponding energy levels, $1s, 2s, 3s, \cdots$ are given by

$$E_{no} = \frac{\hbar^2}{2M}\left(\frac{n\pi}{R}\right)^2 = E_{ns} \tag{8.5.4}$$

For $l = 1$ (p-state), the radial equation is

$$-\frac{\hbar^2}{2M}\cdot\frac{1}{r}\frac{d^2}{dr^2}(ru_{nl}) + \frac{\hbar^2}{Mr^2}u_{nl} = Eu_{n1}$$

and the solution, finite at $r = 0$, is given by

$$u_{n1} = \frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}; \quad E_{n1} = -\frac{\hbar^2}{2M}k^2$$

k is given by the nodes (roots) of the equation $u_{n1}(R) = 0$.
Let $kR = x$, then the values of x for which

$$u_{n1}(R) = 0 \text{ are } x_{1p} = 4.49, \quad x_{2p} = 7.73$$

The corresponding energies are given by

$$E_{n1} = \frac{\hbar^2}{2M}\left(\frac{x_{n1}}{R}\right)^2 = E_{np} \tag{8.5.5}$$

The energy levels are $1p, 2p, 3p, \cdots, n$ is thus *not* the principal quantum number, as stated already.

$u_{no}(r)$, $u_{n1}(r)$ are the spherical Bessel functions

$$J_1(kr) = r^{-\frac{1}{2}} J_{1+\frac{1}{2}}(kr)$$

Thus, for l; the energy levels are given by

$$\boxed{E_{nl} = \frac{\hbar^2}{2M}\left(\frac{x_{nl}}{R}\right)^2} \tag{8.5.6}$$

and x_{nl} are available in Tables of standard mathematical functions.

For each E_{nl}, there are $(2l+1)$ allowed values of quantum number m. Since coupling of spin and orbital motion of a nucleon is neglected, each nucleon has two possible spin states : $m_s = +\frac{1}{2}$, $m_s = -\frac{1}{2}$. We thus have $2(2l+1)$-fold degeneracy for each E_{nl}.

Spin-orbit coupling — Next, we consider the spin-orbit coupling term in the self-consistent potential (in which each nucleon is supposed to move) of the form

$$V_{so}(\vec{r})\,\vec{l}\cdot\vec{s}$$

It can be shown that l and s are good quantum numbers but m and m_s are no longer good. Thus the magnitude but not the directions of $\vec{l}$ and $\vec{s}$ are preserved. The total angular momentum $j = l + s$ is a conserved quantity and the states will be characterised by (l, s, j, j_z). So for a given l and $s = \frac{1}{2}$, there are two j-values $\left(l+\frac{1}{2}\right)$ and $\left(l-\frac{1}{2}\right)$. The allowed values of j_z are $2l+2$ and $2l$ respectively.

The introduction of spin-orbit coupling thus split $2(2l+1)$-fold degeneracy levels into two, levelled by $(nl)_{l+\frac{1}{2}}$ and $(nl)_{l-\frac{1}{2}}$ when $l = 2$ (d-states).

$$nd \text{ (10 states)} \longrightarrow nd_{5/2} \text{ (6 states)} + nd_{3/2} \text{ (4 states)}$$

In the following figure, the orbits in each shell are specified where n stands for the number assigned to the shell, the other notations have their usual meanings. The corresponding energy level diagram in the shell theory potential is also shown (Fig. 8.12).

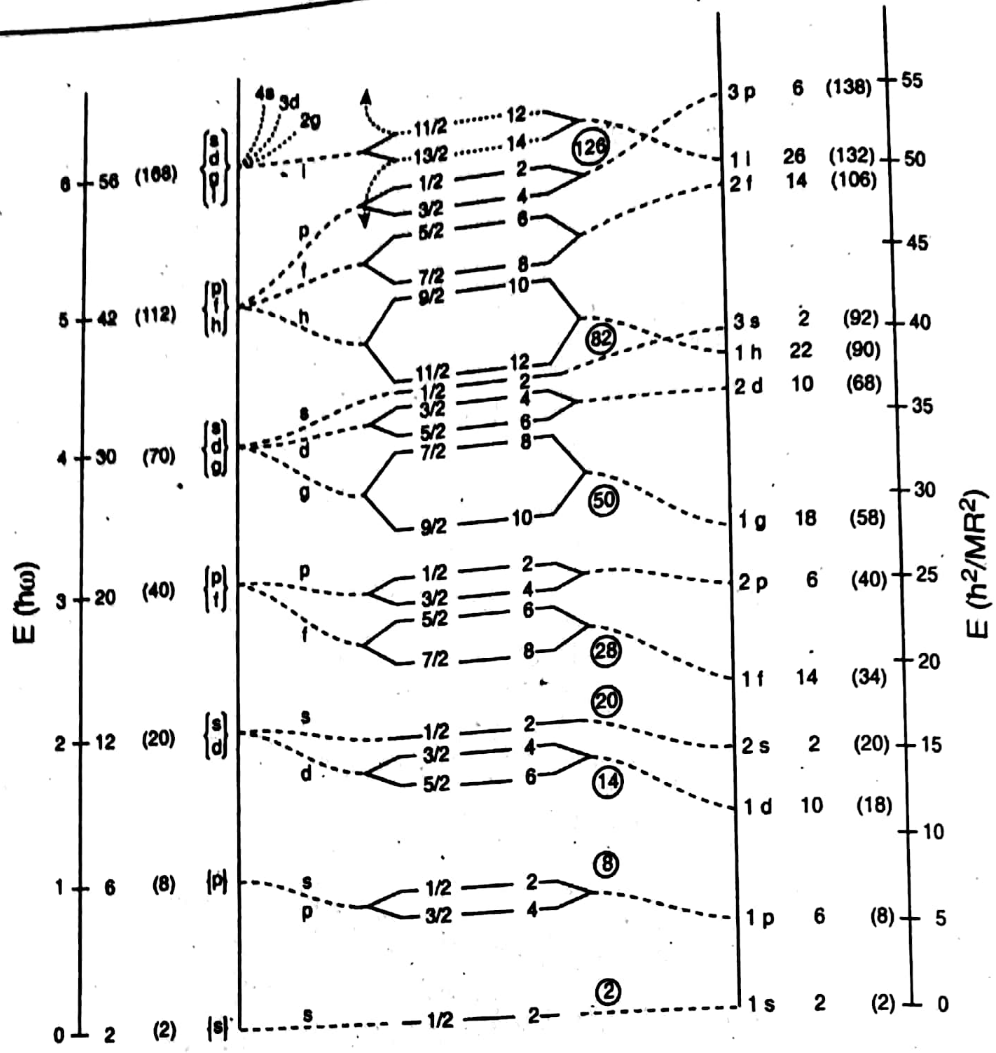

Fig. 8.12 Nuclear energy levels given by the shell model. On the left we have harmonic oscillator levels; at the right the square well-bound states. Spin-orbit coupling effect is shown in the middle, which shows the emergence of magic numbers

Table 8.3 : *Orbits in each shell*

$l \rightarrow$ $n \downarrow$	s 0	p 1	d 2	f 3	g 4	h 5	i 6
1	$1s$						
2		$1p$					
3	$2s$		$1d$				
4		$2p$		$1f$			
5	$3s$		$2d$		$1g$		
6		$3p$		$2f$		$1h$	
7	$4s$		$3d$		$2g$		$1i$

Each state in the above diagram (Fig. 8.12) represents $(2j + 1)$-allowed orbits corresponding to different values of n. Thus, the total number of orbits in $n = 1$ is 2; in $n = 2$ is $4 + 2 = 6$; in $n = 3$ is $6 + 2 + 4 + 8 = 20$; in $n = 4$ is $4 + 6 + 2 + 10 = 22$ and $n = 5$ is $6 + 8 + 2 + 4 + 12 = 32$ etc.